Numerical Solutions of a Projectile Motion Model

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Outline

- Introduction
- Defining and Solving the Problem
- Fixed Points and Iterative Methods
- Inverse and Optimization Problem
- Numerical Algorithms and Results
- Conclusion

Inverse Problem

- What is an Inverse Problem?

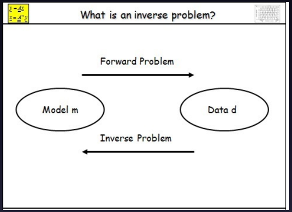
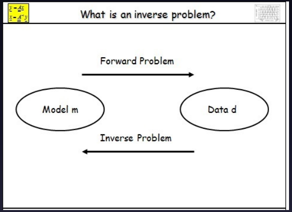


Figure : Inverse Problems

Inverse Problem

- What is an Inverse Problem?
- What do they Influence?



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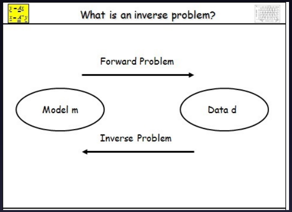

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Techniques Needed

- Differential Equations
  — To build model representing projectile motion
- Fixed Points and Fixed Point Iteration
  — Numerically solve implicitly defined model
- Optimization
  — Optimize the possible range
- Numerical Methods
  — Solve inverse optimization problem numerically

Defining the Problem

- Suppose we launch a point projectile from the origin with
  — Initial angle θ (radians)
  — Initial velocity v (feet/second)
  — Unit mass (1 gram)

- The projectile is then subject to
  — Air resistance with coefficient k
  — Gravitational force $g = -32\ (ft/sec^2)$

- The total forces can thus be represented by

$$-k \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} + \begin{pmatrix} 0 \\ -g \end{pmatrix} \tag{1}$$

Projectile Motion

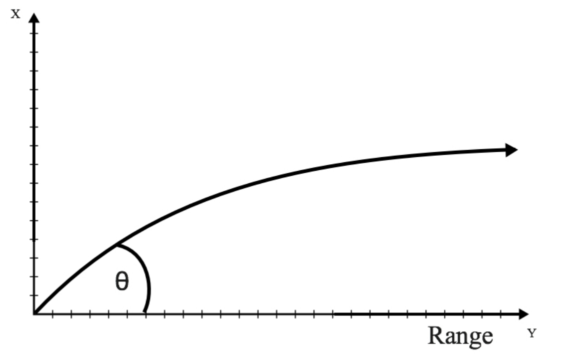

Figure : Graph of Projectile Motion

Initial Value Problems

- We can develop a system of two initial value problems (IVPs) to represent the motion of the projectile.

$$\begin{aligned} \ddot{x} &= -k\dot{x} \\ \dot{x}(0) &= v\cos\theta \\ x(0) &= 0 \end{aligned} \tag{2}$$

and

$$\begin{aligned} \ddot{y} &= -k\dot{y} - g \\ \dot{y}(0) &= v\sin\theta \\ y(0) &= 0 \end{aligned} \tag{3}$$

Solving the Problem

- Solving the initial value problems through basic substitution methods, we reach

$$x = \frac{v\cos\theta(1-e^{-kt})}{k} \tag{4}$$

$$y = \left(\frac{v\sin\theta}{k} + \frac{g}{k^2}\right)(1-e^{-kt}) - \frac{g}{k}t \tag{5}$$

Solving the Problem Cont'd

- Solving (4) for t we have,

$$t = -\frac{1}{k} ln\left(1 - \frac{ks}{v\cos\theta}\right) \quad (6)$$

substituting (6) into (5) and simplifying we have

$$y = x\left(\frac{vsin\theta}{k} + \frac{g}{k^2}\right)\left(\frac{kx}{vcos\theta}\right) + \frac{g}{k^2} ln\left(1 - \frac{kx}{vcos\theta}\right) \quad (7)$$

Thus we know x is a root of the equation (7). We then have,

$$x = \frac{vcos\theta}{k}\left(1 - e^{-\left(\frac{k}{v}sec\theta + \frac{k^2}{g}tan\theta\right)x}\right) \quad (8)$$

Defining Range Function

- The range equals the distance moved in the x direction, thus we can see that $x = R(\theta)$ is a root of

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- where

$$A(\theta) = a\sec\theta + b\tan\theta$$
$$a = \frac{k}{v} \; and \; b = \frac{k^2}{g}, \quad a > 0, \; b > 0.$$

Non-Implicit Functional

- In (9) the range value, $R(\theta)$, is defined implicitly. It can be written in equivalent functional form

$$F(\theta, r) = \frac{cos\theta}{a}\left(1 - e^{1A(\theta)r}\right), \quad r > 0 \;\&\; \theta \in \left[0, \frac{\pi}{2}\right] \quad (10)$$

- For future reference, note
— $a, A(\theta)$ are as defined above
— $\theta \in \left[0, \frac{\pi}{2}\right]$ implies $\frac{cos\theta}{a} > 0$ and $\frac{cos\theta A(\theta)}{a} > 1$
— $F_r(\theta, r)$ and $F_\theta(\theta, r)$ exist and are continuous
— $F(\theta, r)$ is classically differentiable and thus continuous on $\left[0, \frac{\pi}{2}\right]$

Fixed Points

Definition

A fixed point of a function f is defined as a point p such that $f(p) = p$.

- Example: $f(x) = x^2$ has two fixed points $x = 0$ and $x = 1$
- Graphically, fixed points of a function are intersections between that function and the line $y = x$

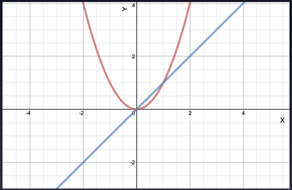

Figure : Graph of $y = x^2$ and $y = x$

Fixed Points of the Functional

- To study the fixed points of functional (10) we work with a simplified, but equivalent form. Let

$$f(x) = C\left(1 - e^{-dx}\right), \quad C > 0,\ Cd > 1,\ \&\ x > 0 \qquad (11)$$

where $C = \frac{cos\theta}{a}$ and $d = A(\theta)$.

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where $C = \frac{cos\theta}{a}$ and $d = A(\theta)$.

- It can easily be shown that
— 0 is a fixed point of f, by definition
— For sufficiently small s, $f(s) > s$, proof using L'Hopitals Rule
— $f(C) < C$ for C defined as above, from conditions on C

Fixed Points of the Functional Cont'd

- Since f is continuous, by the Intermediate Value Theorem, there exists a point, $p \in (0, C)$, such that $f(p) = p$. Thus, by definition, p is a fixed point of f.

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- It is easily shown that the second derivative of f is strictly negative and thus f is concave down and thus the graph can intersect the line $y = x$ at a maximum of two points in the domain. Since 0 is a known fixed point, we conclude p is a unique positive fixed point.
- Furthermore, it can be shown that if $f(x) > x$, then $x < p$ and consequently $f(x) < x \implies x > p$ for all $x \geq 0$. The proof of this follows from p being unique.

Iterative Methods

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- The results found while studying fixed point iteration with equation (11) can be applied to (10). From this we conclude that $R(\theta)$ is the unique positive fixed point of $F(\theta, r)$ and the fixed point iteration is a suitable method of solving the implicitly defined functional in (9).

Inverse Problem

- We work with solving the inverse problem of finding the angle at which a projectile should be launched to reach a suboptimal range. We define

$$g(t) = at = 1 + e^{-(at+b\sqrt{t^2-R(\theta)^2})} \tag{12}$$

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- Note: $R(\theta)$ is a solution of equation (10) if and only if $t = R(\theta)\sec(\theta)$ is a root of the function $g(t)$ defined in (12). Proof:

$$\begin{aligned} & R(\theta) = \frac{\cos\theta}{a}(1 - e^{-A(\theta)R(\theta)}) \\ \implies \quad & a\sec(\theta)R(\theta) = 1 - e^{-(a\sec\theta + b\tan\theta)R(\theta))} \\ \implies \quad & at = 1 - e^{-(at+b\tan\theta R(\theta)} \\ \implies \quad & 0 = at - 1 + e^{-(at+b\sqrt{t^2 - R(\theta)^2})} \\ \implies \quad & g(t) = 0 \end{aligned} \tag{13}$$

The converse can be proved in similar fashion.

Optimization

- We also develop the inverse problem of finding the angle corresponding to the maximum range. Note that the second partial derivative is negative, thus critical points are maximums

$$\begin{aligned}
&\frac{\cos\theta}{a}\left(A'(\theta)R(\theta)\right)e^{-A(\theta)R(\theta)} - \frac{\sin\theta}{a}\left(1 - e^{-A(\theta)R(\theta)}\right) = 0 \\
&R(\theta)\left[\tan\theta\left(e^{-A(\theta)R(\theta)} - 1\right) + c\ \sec\theta e^{-A(\theta)R(\theta)}\right] = 0,\ c = \frac{b}{a} \\
&\sin\theta - \sin\theta e^{-A(\theta)R(\theta)} - c\ e^{-A(\theta)R(\theta)} = 0 \\
&\sin\theta = (\sin\theta + c)e^{-A(\theta)R(\theta)} \qquad (14)
\end{aligned}$$

Optimization Cont'd

- Taking arc sine on both sides, which exists since $\theta \in \left(0, \frac{\pi}{2}\right)$, we can find θ the solution of the inverse problem. In order to compute the angle we must find an equivalent form that is suitably defined.
  From (10) we can see

$$e^{-A(\theta)R(\theta)} = 1 - a \sec\theta R(\theta) \tag{15}$$

Substituting (15) into (14) we have

$$\begin{aligned} & \sin\theta = (\sin\theta + c)(1 - a\sec\theta R(\theta)) \\ \implies \quad & R(\theta) = \frac{(c/a)\cos\theta}{\sin\theta + c} \\ \implies \quad & A(\theta)R(\theta) = \frac{c + c^2\sin\theta}{\sin\theta + c} \\ \implies \quad & \sin\theta = (\sin\theta + c)e^{-\left(\frac{c+c^2\sin\theta}{\sin\theta+c}\right)} \end{aligned} \tag{16}$$

Numerical Algorithms

- For the Direct Problem, we solve our implicitly defined equation (9) using the fixed point iteration method.

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- For the Inverse Problem, equation (16) can be written in the equivalent form

$$x = e^{hx}, \quad x = \frac{e \sin\theta}{\sin\theta + c} \quad \& \quad h = \frac{1-c^2}{e} \tag{17}$$

The numerical algorithm then solves equation (17) using Newton's Method, setting $\sin\theta = \frac{cx}{e-x}$ and $\theta = \sin^{-1}\left(\frac{cx}{e-x}\right)$.

Results — Direct Problem

- Solutions of the direct problem using fixed point iteration.

θ	R	V	R
$\pi/12$	70.88511102176	100	67.34060878040
$2\pi/12$	77.88306236704	300	2.12024170366 $\times 10^2$
$3\pi/12$	67.34060878040	500	3.53551174056$\times 10^2$
$4\pi//12$	48.61653757497	700	4.94974708427 $\times 10^2$
$5\pi/12$	25.36860773980	900	6.36396102456 $\times 10^2$
$6\pi/12$	6.01470426990 $\times 10^{-15}$	1100	7.77817459295 $\times 10^2$

Table : Values of range for varying values of speed and initial angle with fixed k=1

Results — Direct Problem Cont'd

- The range values computed numerically based on the direct problem.

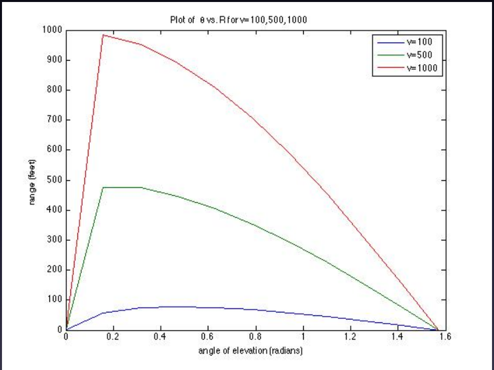

Figure : Plot of theta vs. range for varying values of speed: v=100,500,1000 and k=1

Results — Inverse Problem

- The following tables compare the angles for varying values of speed, which produce the maximum range.

V	θ
100	0.459362551800941
200	0.347971552133387
300	0.286456352026570
400	0.246237533256279
500	0.217435525427001
600	0.195582070744295
700	0.178320450757590
800	0.164274857263486
900	0.152581613689122
1000	0.142668003658631

Table : Angles which produce the optimum range for varying values of speed

Results — Inverse Problem Cont'd

- In the following figure, the speed starts at $V = 100$ ft./sec. and is incremented by 10. The graph plots the number of increments along the x-axis and the value of theta along the y-axis.

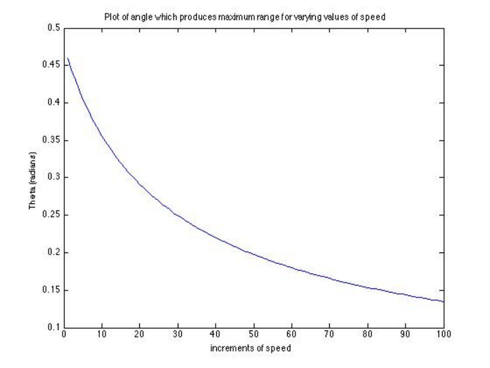

Figure : Increments of speed vs. value of theta that produces maximum range

Conclusion

- We modeled the range of a point projectile as a function of the angle of elevation based on scientific knowledge.
- We defined the initial conditions and equations of motion to reflect the air resistance on the projectile using trigonometry.
- We then studied fixed points and fixed point iteration, and used iterative methods to numerically solve the equation.
- We solved the inverse problem of finding the angle that produces either the maximum range or a given suboptimal range.
- We showed that the iteration sequence converges monotonically to the fixed point for any positive initial guess, this helps ensure numerical stability.
- We analyzed the relationship between the initial speed, the angle of elevation, and the range.

References

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Thank You

Questions?